The p-adic Γ-function

Sidi Mohammed Boughalem (shinokiz@gmail.com)

19. July 2019

The Gamma function Γ is one commonly used extension of the factorial function to the reals (and even complex numbers). In this talk, we will introduce its p-adic analog according to Morita [1], and prove some of its properties. We will first recall the real Gamma function through Euler's integral, present some properties and show how we can extend it continuously into a function on $\mathbb{Z}_p$ through interpolation.

1 The real Γ-function

Proposition 1.1 (Euler's integrals of the 2nd kind)**.**

$$\int_0^{+\infty} e^{-t} t^{x-1} dt$$

exists for all $x \in \mathbb{R}_{>0}$

Proof. We will proceed in three steps, consider

$$\int_0^{+\infty} e^{-t} t^{x-1} dt = \int_0^1 e^{-t} t^{x-1} dt + \int_1^{+\infty} e^{-t} t^{x-1} dt$$

Step 1: $x = n \in \mathbb{N}_{>0}$.

$$\int_0^1 e^{-t} t^{n-1} dt < \infty$$

Since for $0 \leq t \leq 1$ one has that $e^{-t} \leq 1 \Rightarrow e^{-t} t^{n-1} \leq t^{n-1}$ and

$$\int_0^1 t^{n-1} dt = \frac{1}{n}$$

On the other hand,

$$\int_1^{+\infty} e^{-t} t^{n-1} dt < \infty$$

Seminar „p-adic Analysis“, SS 19, Universität Regensburg

Since for $t \geq 1$ one has (by L'Hopital's rule)

$$\lim_{t\to\infty} \frac{t^{n-1}}{e^{\frac{t}{2}}} = \lim_{t\to\infty} \frac{(n-1)t^{n-2}}{\frac{1}{2}e^{\frac{t}{2}}} = \cdots = \lim_{t\to\infty} \frac{(n-1)!}{(\frac{1}{2})^n e^{\frac{t}{2}}} = 0$$

thus for $\epsilon = 1$, $\exists M > 0$ such that for all $t > M$

$$t^{n-1} \leq e^{\frac{t}{2}} \Rightarrow t^{n-1}e^{\frac{t}{2}} \leq e^{-\frac{t}{2}} \text{ and } \int_1^{+\infty} e^{-\frac{t}{2}}dt = 2e^{-\frac{1}{2}}$$

Step 2: $x \in \mathbb{R}, x \geq 1$. One has $\lfloor x \rfloor \leq x \leq \lfloor x \rfloor + 1$ and for $t \geq 0$

$$e^{-t}t^{x-1} \leq e^{-t}t^{\lfloor x \rfloor} \text{ and } \int_0^{+\infty} e^{-t}t^{\lfloor x \rfloor}dt < \infty \text{ (by Step 1)}$$

Step 3: $x \in \mathbb{R}, 0 < x < 1$. We distinguish between two cases.

1. If $0 \leq t \leq 1$

$$e^{-t} \leq 1 \Rightarrow e^{-t}t^{x-1} \leq t^{x-1} \text{ and as } \int_0^1 t^{x-1}dt = \frac{1}{x}$$

$$\Rightarrow \int_0^1 e^{-t}t^{x-1}dt < \infty$$

2. If $t \geq 1$

$$1 \leq t^{x-1} \leq t \Rightarrow e^{-\frac{t}{2}} \leq t^{x-1}e^{-\frac{t}{2}} \leq te^{-\frac{t}{2}}$$

As both

$$e^{-\frac{t}{2}}, te^{-\frac{t}{2}} \xrightarrow[t\to\infty]{} 0 \text{ (\textit{L'Hopital} again for example)}$$

$$\Rightarrow t^{x-1}e^{-\frac{t}{2}} \xrightarrow[t\to\infty]{} 0 \forall x \in]0,1[$$

Again, $t^{n-1}e^{\frac{t}{2}} \leq e^{-\frac{t}{2}}$ and $\int_1^{+\infty} e^{-\frac{t}{2}}dt = 2e^{-\frac{1}{2}}$, hence

$$\int_1^{+\infty} e^{-t}t^{x-1}dt < \infty$$

□

Now, we are able to define the Γ-function

Definition 1.2. *For all $x \in \mathbb{R}_{>0}$, we define the Γ-function to be*

$$\Gamma(x) := \int_0^{+\infty} e^{-t}t^{x-1}dt$$

We have the following remarkable property:

Proposition 1.3. *For all $x \in \mathbb{R}_{>0}$,*

$$\Gamma(x+1) = x\Gamma(x)$$

In particular, the Γ*-function extends the factorial $n \mapsto n!$ to the positive reals.*

Proof. An integration by parts gives

$$\Gamma(x+1) = \int_0^{+\infty} t^x e^{-t} dt = [-t^x e^{-t}]_0^\infty + \int_0^{+\infty} x t^{x-1} e^{-t} dt$$

$$= x \int_0^{+\infty} t^{x-1} e^{-t} dt = x\Gamma(x)$$

As

$$\Gamma(1) = \int_0^{+\infty} e^{-t} dt = [-e^{-t}]_0^\infty = 1$$

One gets that for $x = n \in \mathbb{N}$

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n!\Gamma(1) = n!$$

□

As $\mathbb{N}$ is not dense in $\mathbb{R}$, there are many ways one could define the factorial that would agree on the integers and disagree elsewhere. But the Γ-function happen to have a property that would make it "the right one": It is the only log-convex function that agrees with factorial on the non-negative integers. A function f is said to be *Log-convex* if its composition with the logarithm, $log \circ f$, is a convex function. Log convexity seems to be a natural property for a function generalizing the factorial, see [2].

Among the many applications of the Γ-function is in the study of the Riemann zeta function. A fundamental property of the Riemann zeta function is its functional equation:

$$\Gamma(\frac{s}{2})\zeta(s)\pi^{-\frac{s}{2}} = \Gamma(\frac{1-s}{2})\zeta(1-s)\pi^{-\frac{1-s}{2}}$$

where it specifies the analytic continuation of the zeta function to a meromorphic function in the complex plane, see [3]. We will cite the following result

Proposition 1.4. *The* Γ*-function satisfies the Legendre relation*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

Proof. a proof can be found in [4]§20, Theorem 5. □

2 the p-adic Γ-function, $p \neq 2$

In this section, p denotes a prime number $\neq 2$. We will treat the case $p = 2$ in the next section. Recall that $\mathbb{N}$ is dense in $\mathbb{Z}_p$. Given a sequence $(a_n)_{n\geq 0}$ in some field K, there exist at most one continuous function $f : \mathbb{Z}_p \longrightarrow K$ such that $f(n) = a_n$ for all $n \in \mathbb{Z}_p$.

Proposition 2.1. *a sequence $(a_n)_{n\geq 0}$ in K can be interpolated if*

$$\exists f : \mathbb{Z}_p \longrightarrow K \text{ such that } f(n) = a_n \text{ for all } n \in A,\ \overline{A} = \mathbb{Z}_p.$$

$$\Leftrightarrow\ \forall \epsilon > 0\ \exists N > 0 \text{ such that for all } n = m + p^N,\ |f(n) - f(m)| < \epsilon$$

In other words, if two integers m, n differ by a large power of p, then $f(n) - f(m)$ gets smaller. Now if $a_n \in \mathbb{Z}$ with $K = \mathbb{Q}_p$, we want to find a function $f : \mathbb{Z}_p \longrightarrow \mathbb{Q}_p$ such that

$$f(n+1) = n!\ \forall n \in \mathbb{N} \tag{1}$$

2.1 Naive approach

At first, one might simply think of taking the function $n \mapsto n!$ and extend continuously to $\mathbb{Z}_p$. Indeed suppose there exist a function verifying (1). By continuity and density of $\mathbb{N}$ in $\mathbb{Z}_p$, the same holds for all $n \in \mathbb{Z}_p$. Thus, for a fixed power of a prime p^r

$$f(n) = n(n-1)\dots p^r f(p^r - 1)\ \forall n > p^r$$

As $\mathbb{Z}_p$ is compact, $\exists C > 0$ such that $|f(x)|_p \leq C$ for all $n \in \mathbb{Z}_p$. In particular, for $n > p^r$

$$|f(n)|_p = |n(n-1)\dots p^r f(p^r - 1)|_p \leq |p|_p^r C$$

By density,

$$||f||_\infty \leq |p|_p^r C$$

If r=1, then if $||f||_\infty = C$, one has

$$||f||_\infty \leq |p|_p ||f||_\infty \Rightarrow ||f||_\infty (1 - \frac{1}{p}) \leq 0 \Rightarrow ||f||_\infty = 0$$

Hence the only continuous function $f : \mathbb{Z}_p \longrightarrow \mathbb{Q}_p$ satisfying (1) is $f = 0$. The issue here is that

$$p \mid (n!)$$

We need hence to slightly modify the factorial by considering a "restricted" version of it.

2.2 The restricted factorial

Consider the "restricted" factorial defined for all $n \in \mathbb{N}$

$$n!^* := \prod_{\substack{0 \le j < n \\ p \nmid j}} j \tag{2}$$

Theorem 2.2 (Willson's congruence)**.** *Let $n > 1$, $n \in \mathbb{N}$*

$$(n-1)! \equiv -1 \mod n \Leftrightarrow n \textit{ is prime.}$$

Proof. If $n \geq 3$, n is not a prime, then $\exists q \in \{2, \dots, n-2\}$ prime such that $q \mid n$. Hence

$$(n-1)! \equiv -1 \mod n \Rightarrow (n-1)! \equiv -1 \mod q$$

which is impossible since $q \mid (n-1)!$ □

We need a generalization of this result:

Theorem 2.3 (Willson's generalized congruence)**.** *Let $a \in \mathbb{N}$, $r \geq 1$ and $p \geq 3$*

$$\prod_{\substack{a \le j < a+p^r \\ p \nmid j}} j \equiv -1 \mod p^r$$

Proof. Let $a \leq j < a + p^r$ and suppose $p \nmid j$. The set $\{a, a+1, \dots, a+p^r-1\}$ forms a complete set of representatives modulo $p^r\mathbb{Z}$ where the elements j such that $p \nmid j$ represent the invertible elements. Consider the morphism $\varphi : \mathbb{Z} \twoheadrightarrow \mathbb{Z}/_{p^r}$

$$\varphi(\prod_{\substack{a \le j < a+p^r \\ p \nmid j}} j) = \prod_{j \in (\mathbb{Z}/p^r)^\times} j = \prod_{j = j^{-1}} j = \prod_{j^2=1} j$$

since all the other terms $j \neq j^-1$ cancel in the product. Recall that $\mathbb{U} := (\mathbb{Z}/_n)^\times$ is cyclic for $n = p^r$, p odd, thus it contains a unique non-trivial subgroup of order 2 and thus

$$\prod_{j^2=1} j = 1 \times -1 = -1$$

and hence

$$\prod_{\substack{a \le j < a+p^r \\ p \nmid j}} j \equiv -1 \mod p^r$$

□

Now back to (2), for $m = n + p^r$

$$(n+p^r)!^* = \prod_{\substack{0 \le j < n+p^r \\ p \nmid j}} j = (\prod_{\substack{0 \le j < n \\ p \nmid j}} j)(\prod_{\substack{n \le j < n+p^r \\ p \nmid j}} j) = n!^*(\prod_{\substack{n \le j < n+p^r \\ p \nmid j}} j) \equiv -n!^* \mod p^r$$

By adjusting one last time the sign, we define the function

$$f(n) := (-1)^n \prod_{\substack{0 \leq j < n \\ p \nmid j}} j \tag{3}$$

Clearly, for $m \equiv n \mod p^r$ one has

$$f(m) = f(n + p^r) = (-1)^{n+p^r} \prod_{\substack{0 \leq j < n+p^r \\ p \nmid j}} j = [(-1)^n \prod_{\substack{0 \leq j < n \\ p \nmid j}} j][(-1)^{p^r} \prod_{\substack{n \leq j < n+p^r \\ p \nmid j}} j]$$

$$\equiv (-1)^{p^r+1} f(n) \equiv f(n) \mod p^r \text{ (as p is odd).}$$

Hence

$$|f(m) - f(n)|_p = p^{-r} \xrightarrow[r \to \infty]{} 0$$

Thus the function $f : \mathbb{N}_{\geq 2} \to \mathbb{Z}$ can be continuously extended into a function $\mathbb{Z}_p \to \mathbb{Z}_p$.

Definition 2.4 (Morita's p-adic Γ-function)**.** *The Morita p-adic Γ-function is the continuous function*

$$\Gamma_p : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p^\times$$

extending

$$f(n) := (-1)^n \prod_{\substack{0 \leq j < n \\ p \nmid j}} j, \ \ n \geq 2$$

Note that the following holds for $n \in \mathbb{N}$

$$\Gamma_p(n+1) = \begin{cases} n! & \text{if } n \text{ odd, } n \leq p-1 \\ -n! & \text{if } n \text{ even, } n \leq p-1 \end{cases} \tag{4}$$

And

$$\Gamma_p(n+1) = \begin{cases} -n\Gamma_p(n) & \text{if } p \nmid n \\ -\Gamma_p(n) & \text{if } p \mid n \end{cases} \tag{5}$$

By continuity and density, one has

$$\Gamma_p(x+1) = \begin{cases} -x\Gamma_p(x) & \text{if } x \in \mathbb{Z}_p^\times \\ -\Gamma_p(x) & \text{if } x \in p\mathbb{Z}_p \end{cases}$$

Pose

$$h_p(x) = \begin{cases} -x & \text{if } |x|_p = 1 \\ -1 & \text{if } |x|_p < 1 \end{cases}$$

Then we get finally

$$\Gamma_p(x+1) = h_p(x)\Gamma_p(x). \tag{6}$$

We summarize the above results into the following theorem:

Theorem 2.5. *For p an odd prime, $\Gamma_p : \mathbb{Z}_p \longrightarrow \mathbb{Q}_p$ is continuous, $\Gamma_p(\mathbb{Z}_p) \subset \mathbb{Z}_p^\times$. Moreover one has for all $x, y \in Z_p$:*

1. $\Gamma_p(0) = 1, \Gamma_p(1) = -1, \Gamma_p(2) = 1, \Gamma_p(3) = -2, \ldots$
2. $|\Gamma_p(x)|_p = 1$.
3. $|\Gamma_p(x) - \Gamma(y)|_p \leq |x - y|_p$.
4. $\Gamma_p(x)\Gamma_p(1-x) = (-1)^{l(x)}$

$$\begin{aligned} l : \mathbb{Z}_p &\to \{0, \ldots, p-1\} \\ x &\mapsto x \mod p\mathbb{Z}_p \end{aligned}$$

The three first properties are direct results of what we have seen above, we take a deeper look at (4).

(4). By density again, we only need to prove it for $n \in \mathbb{N}$. Let

$$f(n) = \Gamma_p(n)\Gamma_p(1-n)$$

One has

$$f(n+1) = h_p(n)\Gamma_p(n)\Gamma_p(-n) = h_p(n)\Gamma_p(n)\frac{\Gamma_p(1-n)}{h_p(-n)} = \epsilon(n)f(n)$$

Where

$$\epsilon(n) = \frac{h_p(n)}{h_p(-n)} = \begin{cases} -1 & \text{if } p \mid n \\ 1 & \text{if } p \nmid n \end{cases}$$

Hence

$$f(n+1) = \epsilon(n)f(n) = \cdots = (-1)^k f(1) = (-1)^{k+1}$$

where

$$k = \#\{0 \leq j < n, p \mid j\} = n - [n/p]$$

Now let $m = n + 1$, then $1 - m = -n$ and

$$\Gamma_p(m)\Gamma_p(1-m) = \Gamma_p(n+1)\Gamma_p(-n) = (-1)^{n-[n/p]+1}$$

Let $n = n_0 + n_1 p + \cdots = n_0 + p[n/p]$, then one has $n - [n/p] = n_0 + (p-1)[n/p]$. As p is odd, $p - 1$ must be even and thus $n - [n/p]$ has the same parity as n_0. Hence

$$\Gamma_p(m)\Gamma_p(1-m) = (-1)^{n-[n/p]+1} = (-1)^{n_0+1}$$

as $m = n + 1 \equiv n_0 + 1 \mod p$, $n_0 + 1 \in \{0, \ldots, p-1\}$ and $n_0 + 1 = l(m)$. □

Recall Proposition (1.4) gives for $z = \frac{1}{2}$:

$$\Gamma(\frac{1}{2})^2 = \pi \text{ and } \Gamma(\frac{1}{2}) = \sqrt{\pi}$$

Hence, in $\mathbb{Q}_p$, the analogue of π can be taken to be

$$\Gamma_p(\frac{1}{2})^2 = (-1)^{l(\frac{1}{2})}$$

where $l(\frac{1}{2}) = l(\frac{p+1}{2}) = \frac{p+1}{2}$ hence

$$\Gamma_p(\frac{1}{2})^2 = (-1)^{\frac{p+1}{2}} = \begin{cases} 1 & \text{if } p \equiv 3 \mod 4 \\ -1 & \text{if } p \equiv 1 \mod 4 \end{cases}$$

Thus if $p \equiv 1 \mod 4$, $\Gamma_p(\frac{1}{2})$ is a canonical square root of -1 and one has

$$\Gamma_p(\frac{1}{2}) \equiv \Gamma_p(\frac{p+1}{2}) = (-1)^{\frac{p+1}{2}} \prod_{\substack{0 \leq j < \frac{p+1}{2} \\ p \nmid j}} j = (-1)^{\frac{p+1}{2}} (\frac{p+1}{2})! \equiv -(\frac{p+1}{2})! \mod p$$

3 Save the 2-adic Γ-function!

In this section we focus solely on the case where $p = 2$. As seen before, most of the results that we used in order to prove the interpolation property of the factorial relied on results that worked only for odd primes. We will thus reproduce the same construction with the same arguments using other tools this time. But first we prove the following result that will make up for Willson's generalized theorem (2.3):

Proposition 3.1. *For $r \geq 3$ we have $(\mathbb{Z}/2^r)^\times \cong \mathbb{Z}/2 \times \mathbb{Z}/2^{2r-2}$*

Proof. Consider the morphism

$$\phi : (\mathbb{Z}/2^r)^\times \longrightarrow (\mathbb{Z}/4)^\times \cong \mathbb{Z}/2$$
$$x \longmapsto x \mod 4$$

As $|(\mathbb{Z}/2^r)^\times| = \varphi(2^r) = 2^{r-1}(2-1) = 2^{r-1}$, $|\ker(\phi)| = 2^{r-2}$ since $\ker(\phi) \cap (\mathbb{Z}/2) = \{1\}$. Hence we only need to show that $\ker(\phi)$ has an element of order 2^{r-2}. We proceed by induction: For $r = 3$ it is trivial, suppose now that

$$5^{2^{r-2}} \equiv 1 \mod 2^r$$

then the order of 5 modulo 2^{r+1} $o(5) = k2^{r-2}$ is a multiple of 2^{r-2}, for $k = 1, 2$ or 4. If $k = 4$ then $o(5) = k2^r$ which is the order of the group, but since

$$\begin{cases} (2^r - 1)^2 = 2^{2r} - 2^{r+1} + 1 & \equiv 1 \mod 2^r \\ (2^{r-1} - 1)^2 = 2^{2r-2} - 2^r + 1 & \equiv 1 \mod 2^r \end{cases}$$

these two elements of order 2 generate two different subgroups, thus $k = 1, 2$. Since $5^{2^{r-2}} \equiv 1 \mod 2^r$ one has that

$$\begin{aligned} & 5^{2^{r-3}} \equiv 1 && \mod 2^{r-1} \\ \Rightarrow\ & 5^{2^{r-3}} \equiv 1 + k2^{r-1} && \mod 2^r \text{ for } k = 0, 1. \end{aligned}$$

if $k = 0$, then we would have that $5^{2^{r-3}} \equiv 1 \mod 2^{r-1} \Rightarrow 2^{r-2} \mid 2^{r-3}$ which cannot be true, hence

$$\begin{array}{lrll}
 & 5^{2^{r-3}} & \equiv 1 + 2^{r-1} & \mod 2^r \\
 \Rightarrow & 5^{2^{r-3}} & \equiv 1 + 2^{r-1} + k2^r & \mod 2^{r+1} \text{ for } k = 0, 1. \\
\text{We square both sides} & 5^{2^{r-3}} & \equiv 1 + 2^{r-1} + k2^r & \mod 2^{r+1} \\
 \Rightarrow & 5^{2^{r-2}} & \equiv (1 + 2^{r-1} + k2^r)^2 & \mod 2^{r+1} \\
 \Rightarrow & 5^{2^{r-2}} & \equiv 1 + 2^{2r-2} + k2^{2r} + 2^r + k2^{r+1} + k2^{r+1} & \mod 2^{r+1} \\
 \Rightarrow & 5^{2^{r-2}} & \equiv 1 + 2^r & \mod 2^{r+1} \\
\text{We square again} & 5^{2^{r-1}} & \equiv (1 + 2^r)^2 & \mod 2^{r+1} \\
 \Rightarrow & 5^{2^{r-1}} & \equiv 1 + 2^{2r} + 2^{r+1} & \mod 2^{r+1} \\
 \Rightarrow & 5^{2^{r-1}} & \equiv 1 & \mod 2^{r+1}
\end{array}$$

Hence, by induction, $\ker(\phi) \cong \mathbb{Z}/_{2^{r-2}}$ and thus the result. □

Now consider

$$f(n) := (-1)^n \prod_{\substack{0 \le j < n \\ j \text{ odd}}} j$$

We know that $f(2k+1) = f(2k)$ and $|f(n)|_2 = 1$. As $Im(f) \subset 1 + 2\mathbb{Z}$ we also have that

$$|f(m) - f(n)|_2 \le \frac{1}{2}$$

Now let $m = n + 2^r$, where $r \ge 3$

$$f(n + 2^r) = (-1)^n \prod_{\substack{0 \le j < n+2^r \\ j \text{ odd}}} j = (-1)^n \prod_{\substack{0 \le j < n \\ j \text{ odd}}} j \prod_{\substack{n \le j < n+2^r \\ j \text{ odd}}} j = f(n) \prod_{\substack{n \le j < n+2^r \\ j \text{ odd}}} j$$

Again, as done before we consider the projection $\varphi : \mathbb{Z} \twoheadrightarrow \mathbb{Z}/_{2^r}$

$$\varphi\Big(\prod_{\substack{n \le j < n+2^r \\ j \text{ odd}}} j\Big) = \prod_{j \in (\mathbb{Z}/2^r)^\times} j = \prod_{j^2 = 1} j$$

From Proposition (3.1) we know that there are no non-trivial cyclic subgroup of order 2 in $(\mathbb{Z}/_{2^r})^\times$ hence

$$\prod_{j^2=1} j = 1 \Rightarrow \prod_{\substack{n \le j < n+2^r \\ j \text{ odd}}} j \equiv 1 \mod 2^r$$

thus

$$|f(m) - f(n)|_2 = \frac{1}{2^r} \xrightarrow[r \to \infty]{} 0$$

We still need to handle the case $r = 2$:

$$f(2n+2) = f(2n+1) = (2n+2)f(2n) = 2nf(2n) + f(2n)$$

$$\Rightarrow |f(2n+2) - f(2n)|_2 = |2nf(2n)|_2 = |2n|_2 \leq |2|_2 = \frac{1}{2}$$

Similarly,

$$\begin{array}{lll} & f(2n+4) & = f(2n+3) = (2n+3)f(2n+2) = 2nf(2n+2) + 3f(2n+2) \\ & & = 4n^2 f(2n) + 8nf(2n) + 3f(2n) \\ \Rightarrow & f(2n+4) - f(2n) & = 4n^2 f(2n) + 8nf(2n) + 2f(2n) \\ & & \equiv 2f(2n) \mod 4 \end{array}$$

Thus

$$|f(2n+4) - f(n)|_2 = |2f(n)|_2 = |2| = \frac{1}{2}$$

As $f([2n+1]+4) - f(2n+1) = f(2n+4) - f(2n)$, one finally has, by density again, that

$$|f(m+4) - f(m)|_2 = \frac{1}{2}$$

Hence, by density and continuity, one finally has that $\forall x, y \in \mathbb{Z}_2$

$$|f(x) - f(y)|_2 \leq |x - y|_2$$

The function f can be continuously extended to the function $\Gamma_2 : \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2^\times \cong 1+2\mathbb{Z}_2$. Hence Definition (2.4) holds for every prime p now. By definition one has similarly as the odd case (6):

$$\Gamma_2(x+1) = h_2(x)\Gamma_2(x). \tag{7}$$

where

$$h_2(x) = \begin{cases} -x & \text{if } x \in 1 + 2\mathbb{Z}_2 \quad (|x|_2 = 1) \\ -1 & \text{if } x \in 2\mathbb{Z}_2 \quad (|x|_2 < 1) \end{cases}$$

Remark 3.2. *Theorem (2.5) holds exactly the same for the case $p = 2$, except a slight change in (3): $\forall x, y \in \mathbb{Z}_2$*

$$|\Gamma_2(x) - \Gamma(y)|_2 \leq k|x-y|_2 \quad \textit{where} \quad k = \begin{cases} 1 & \textit{if } |x-y|_2 < \frac{1}{4} \\ 2 & \textit{if } |x-y|_2 = \frac{1}{4} \end{cases}$$

References

[1] Y. Morita. *A p-adic analogue of the Γ-function*, Journal of the Faculty of Science, the University of Tokyo. Sect. 1 A, Mathematics, 1975.

[2] J. Mollerup, H. Bohr. *Lærebog i Kompleks Analyse*, vol. III, Copenhagen. (Textbook in Complex Analysis) (1922).

[3] B. Riemann. *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsberichte der Berliner Akademie,. November 1859.

[4] O. Forster. *Analysis 1: Differential und Integralrechnung einer Veranderliche*, 10 Aufage, Vieweg+Teubner Verlag, 2011.

[5] W. H. Schikhof. *Ultrametric Calculus: An Introduction to p-adic Analysis*, Cambridge, university press, 1984.

[6] A. M. Robert. *A Course in p-adic Analysis*, (Graduate Texts in Mathematics) Springer (2000).